An Efficient Algorithm For Accurate Reduced Models With The Proper Generalized Decomposition For Magnetoquasi-Static Problems

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***Abstract*—Consideration of electromagnetic field problems solved by the finite-element (FE) method with large FE models yield high computational effort. Model order reduction (MOR) techniques can be applied to reduce the computational effort. The proper generalized decompostition (PGD) is an *a-priori* method which calculates the reduced model (RM) from differential equations separated in previous defined coordinates. To exploit the *a-priori* property of PGD the RM is evaluated for each new mode without the use of a time stepping FE solution. Methods to efficiently evaluate the RM are presented.**

***Index Terms*—Model order reduction, proper generalized decomposition, finite-element method, error estimation.**

I. Introduction

Electromagnetic field problems are time and space dependent. To solve the problem in the space domain the field can be discretized through the finite-element method (FEM). For transient problems a time stepping scheme is used. In case of large problems with a large number of elements in the FE models, the computation time scales with the number of time steps. Model order reduction (MOR) techniques have been successfully applied to reduce the computational effort of such large problems. The MOR techniques can be divided into *a-posteriori*, such as proper orthogonal decomposition [1] and *a-priori* methods. The proper generalized decomposition (PGD) is an *a-priori* method and does not need the evaluation of the reference problem [2]. PGD has been successfully applied to electromagnetic problems [3] [4]. The PGD is divided into an *on-line* and *off-line* process. During the *off-line* process the reduced model (RM) is determined, while in the *on-line* process the solution is calculated from the RM.

In the *off-line* process the model is enriched with modes. The accuracy of the RM strongly depends on the number of modes. To evaluate a sufficient number of modes a reference solution (RS) from a time stepping FEM can be used. In this case the benefit of the *a-priori* approach, that the RS does not need to be calculated is lost. Therefore, efficient error estimation methods to evaluate the RM generated by the PGD without the use of a RS are studied. Nevertheless, the RM is often compared to the RS in terms of magnetic energy or joule and shows good agreement with a small number of modes [5]. This holds for global values, but does not ensure physical correct flux lines in all regions of the model.

II. Proper Generalized Decomposition with A-Formulation

A 3-D magnetic field problem in the space domain Ω with boundary Γ is considered. The definition of a continuous magnetic vector potential $\mathbf{A}(\mathbf{x},t)$ by $\mathrm{curl}\mathbf{A}(\mathbf{x},t) = \mathbf{B}(\mathbf{x},t)$ ensures the condition $\mathrm{div}\mathbf{B} = 0$. Inserting the magnetic vector potential $\mathbf{A}(\mathbf{x},t)$ and the material relationship $\mathbf{H}(\mathbf{x},t) = \nu \cdot \mathbf{B}(\mathbf{x},t)$ into Ampere's and Faraday-Lenz's law yields:

$$\nabla \times (\nu \cdot \nabla \times \mathbf{A}(\mathbf{x},t)) + \sigma \frac{d\mathbf{A}(\mathbf{x},t)}{dt} = \mathbf{J}_s(\mathbf{x},t) \tag{1}$$

$\mathbf{J}_s$ is the source current in a subspace Ω_s and σ the electric conductivity. Applying the weighted residual approach yields the weak formulation of (1)

$$\begin{aligned} &\int_t \int_\Omega \nu \cdot \mathrm{curl}\, \mathbf{A}(\mathbf{x},t) \cdot \mathrm{curl}, \mathbf{a}'(\mathbf{x}) \\ &+ \sigma \partial_t \mathbf{A}(\mathbf{x},t) \cdot \mathbf{a}'(\mathbf{x})\, d\Omega dt \\ &= \int_t \int_{\Omega_s} \mathbf{J}_s(\mathbf{x},t) \cdot \mathbf{a}'(\mathbf{x})\, d\Omega dt \end{aligned} \tag{2}$$

with the linear edge based weighting functions $\mathbf{a}'$ and $\mathbf{H} \times \mathbf{n} = 0$ defined on Γ_H and $\mathbf{B} \cdot \mathbf{n} = 0$ defined on Γ_B with $\mathbf{n}$ the normal unit vector on the boundary . To solve (2) a time stepping technique is commonly applied. In the PGD approach the solution is approximated in a separated form:

$$\mathbf{A}^M(\mathbf{x},t) = \sum_{n=1}^{M} \mathbf{Q}_n(\mathbf{x}) \cdot W_n(t) \tag{3}$$

$\mathbf{Q}_n(\mathbf{x})$ is defined in the space Ω and $W_n(t)$ in the time interval $[0,T]$, M is the number of modes for the approximation $\mathbf{A}^M(\mathbf{x},t)$ with the test function $\mathbf{a}'(\mathbf{x})$ written as

$$\mathbf{a}'(\mathbf{x}) = \mathbf{Q}'_n(\mathbf{x}) \cdot W_n(t) + \mathbf{Q}_n(\mathbf{x}) \cdot W'_n(t) \tag{4}$$

with the test functions $(\mathbf{Q}'_n(\mathbf{x}), W'_n(t))$ defined in the same space as $\mathbf{Q}_n(\mathbf{x}), W_n(t)$ respectively. Since the approximation (3) is separated the source term $\mathbf{J}_s(\mathbf{x},t)$ is separated as well:

$$\mathbf{J}_s(\mathbf{x},t) = \mathbf{J}'(\mathbf{x}) \cdot j_s(t) \tag{5}$$

with $\mathbf{J}'(\mathbf{x})$ a nominal source field and $j_s(t)$ the evolution of the current in time. Supposed the model (3) is already generated to the mode $n-1$, for the next mode $(\mathbf{Q}_n(\mathbf{x}), W_n(t))$ an alternating direction strategy is applied: First $\mathbf{Q}_n(\mathbf{x})$ is assumed to be known and $\mathbf{Q}'_n(\mathbf{x})$ vanishes in (4), resulting in:

$$
\begin{aligned}
&\int_\Omega \nu \cdot \operatorname{curl} \mathbf{Q}_n(\mathbf{x}) \cdot \operatorname{curl} \mathbf{Q}_n(\mathbf{x})\, d\Omega \int_0^T W_n(t) \cdot W_n'(t)\, dt \\
&+ \int_\Omega \sigma \mathbf{Q}_n(\mathbf{x}) \cdot \mathbf{Q}_n(\mathbf{x})\, d\Omega \int_0^T \partial W_n(t) \cdot W_n'(t)\, dt \\
&= \int_\Omega \mathbf{J}'(\mathbf{x}) \cdot \mathbf{Q}_n(\mathbf{x})\, d\Omega \int_0^T j_s(t) \cdot W_n'(t) dt \\
&- \sum_{i=1}^{n-1} \int_\Omega \nu \cdot \operatorname{curl} \mathbf{Q}_i(\mathbf{x}) \cdot \operatorname{curl} \mathbf{Q}_n(\mathbf{x})\, d\Omega \int_0^T W_i(t) \cdot W_n'(t)\, dt \\
&- \sum_{i=1}^{n-1} \int_\Omega \sigma \mathbf{Q}_i(\mathbf{x}) \cdot \mathbf{Q}_n(\mathbf{x})\, d\Omega \int_0^T \partial W_i(t) \cdot W_n'(t)\, dt
\end{aligned}
\tag{6}
$$

Second $W_n(t)$ is assumed to be known and $W_n'(t)$ vanishes in (4) resulting in:

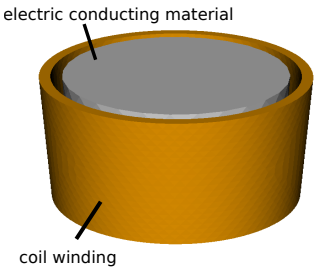
$$
\begin{aligned}
&\int_0^T W_n(t) \cdot W_n(t)\, dt \int_\Omega \nu \cdot \operatorname{curl} \mathbf{Q}_n(\mathbf{x}) \cdot \operatorname{curl} \mathbf{Q}_n'(\mathbf{x})\, d\Omega \\
&+ \int_0^T \partial W_n(t) \cdot W_n(t)\, dt \int_\Omega \sigma \mathbf{Q}_n(\mathbf{x}) \cdot \mathbf{Q}_n'(\mathbf{x})\, d\Omega \\
&= \int_0^T j_s(t) \cdot W_n(t) dt \int_\Omega \mathbf{J}'(\mathbf{x}) \cdot \mathbf{Q}_n'(\mathbf{x})\, d\Omega \\
&- \sum_{i=1}^{n-1} \int_0^T W_i(t) \cdot W_n(t)\, dt \int_\Omega \nu \cdot \operatorname{curl} \mathbf{Q}_i(\mathbf{x}) \cdot \operatorname{curl} \mathbf{Q}_n'(\mathbf{x})\, d\Omega \\
&- \sum_{i=1}^{n-1} \int_0^T \partial W_i(t) \cdot W_n(t)\, dt \int_\Omega \sigma \mathbf{Q}_i(\mathbf{x}) \cdot \mathbf{Q}_n'(\mathbf{x})\, d\Omega
\end{aligned}
\tag{7}
$$

Equation (6) is solved with an euler time-stepping scheme and (7) with the FEM. The iteration stops if the change of $(\mathbf{Q}_n(\mathbf{x}), \cdot W_n(t))$ of two consecutive iteration steps falls under a defined bound.
In the enrichment process the modes are calculated. The stopping criterion of the enrichment process determines the number of modes which are calculated. Possible criterions are the evaluation the newest mode weight compared to the approximation weight or first mode weight:

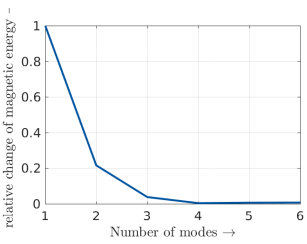
$$
\epsilon(n) = \frac{||\mathbf{Q}_n(\mathbf{x}) \cdot W_n(t)||}{||\sum_{i=1}^{n} \mathbf{Q}_i(\mathbf{x}) \cdot W_i(t)||} \text{ or } \epsilon(n) = \frac{||\mathbf{Q}_n(\mathbf{x}) \cdot W_n(t)||}{||\mathbf{Q}_1(\mathbf{x}) \cdot W_1(t)||}
\tag{8}
$$

To get a stopping criterion with a more physical meaning the change of the mean magnetic energy in the model is calculated for each new mode. In the n^{th} enrichment step the approximation A_d^{n-1} is already calculated an the magnetic energy in the model can be calculated parallel to the new mode. To compare the results with a time stepping FEM ($w_{mag,ref}$) the error is estimated with:

$$
\epsilon(w_{mag}) = \frac{||w_{mag,ref} - w_{mag,pgd}||_2}{||w_{mag,ref}||_2}
\tag{9}
$$



(a) Geometry.



(b) evolution of relative magnetic energy.

Fig. 1: Problems geometry (left) and evolution of relatve magnetic energy change with increasing number of modes.

III. Application

The PGD method is applied to magnetoquasi-static problem. In Fig. 1 the geometry is pictured and consists of a coil winding around an electric conducting material. The winding and conducting material are together discretized with 13951 tetrahedron. The coil is supplied with a sinusoidal current of $I_s = 100\,\mathrm{A}$ resulting in a current density of $J_s = 3.4\,\mathrm{A/m^2}$ and a frequency of $f = 1\,\mathrm{kHz}$. Two periods are simulated with an equidistant discretisation in time of 60 time steps. For every approximation A_d^n the mean magnetic energy is calculated for each time step. Fig. 1b shows the change of the mean magnetic energy with increasing number of modes. From 4 to 6 modes nearly no changes can be observed. A comparison with the reference model shows an error of $\epsilon(w_{mag,n=6}) = 18\,\%$. It is obvious that a pure evaluation of the change in the magnetic energy is not sufficient to estimate the models accuracy without a reference model.

IV. Conclusion

This paper utilizes the proper generalized decomposition method to generate a reduced order model of 3D magnetoquasi-static field problems. The PGD is successfully applied to a simple geometry and analysed in terms of accuracy of the reduced model. Since the energy is a global value no local phenomena are considered so far to evaluate the model in the enrichment process. The full paper will include methods to evaluate the accuracy of the model without a reference solutions. Therefore different methods for error estimation are applied, such as the evolution of the model energy in the first stage of the enrichment process and local gradient methods for higher number of modes.

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